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Cascade Model of Fully Developed Turbulence

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A simple system of quadratically nonlinear equations representing a hierarchy of frequency scales is derived from a cutoff version of Burgers' equation. When forced, the model system reproduces a number of qualitative features of fully developed turbulence. In equilibrium, the model exhibits, in addition to the energy, an extensive isolating integral of the motion that is cubic in the velocity.

KEY WORDS: Intermittency; fully developed turbulence; cascade model; Burgers' equation.

1. INTRODUCTION

Kolmogorov's phenomenological theory for the energy spectrum in a turbulent fluid in three dimensions is perhaps the single most successful theoretical attempt to comprehend fully developed turbulence.(1) Nevertheless it fails to account for certain fluctuation effects ("intermittency") that are seen experimentally in three dimensions.(2) Furthermore, it is formulated in such general terms that it would appear to apply to a number of simpler models, such as Burgers' equation.(3)

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Burgers' equation was originally proposed as a one-dimensional model of the more complicated Navier–Stokes equations. Its "turbulent" or high-Reynold's number solution consists simply of a random array of shocks that yield a k^{-2} energy spectrum, rather than the $k^{-5/3}$ spectrum predicted by Kolmogorov.[3] One expects, on phenomenological grounds, that there are small intermittency-induced deviations from a $-5/3$ exponent even in three dimensions.[4] In this light, Burgers' equation may provide insight into the development and nature of intermittency effects in three dimensions even though it is not quantitatively correct.

An alternative class of models[5–10] of the Navier–Stokes equation has been developed that utilizes only a few degrees of freedom to represent the velocity for each band of wave numbers, $2^n \leqslant k < 2^{n+1}$. These modes are coupled together so as to preserve the relevant conservation laws and the order-of-magnitude interactions between different scales implicit in the Navier–Stokes equations. The energy is observed to cascade through a hierarchy of scales in a manner reminiscent of Kolmogorov's theory. A $-5/3$ (or $-2/3$ when band integrated) energy spectrum is by construction a stationary solution to these models, but in certain cases it is unstable.[9,10] When the resulting temporally intermittent solutions are averaged, statistically stationary correlation functions are obtained that for at least one model prove to be in semiquantitative accord with experiment.[10]

In this paper we will develop another cascade model as a limiting case of Burgers' equation cutoff in Fourier space; that is, the nonlinear convective term is made approximately local in wavenumber space and thus nonlocal in real space. This new model has two real modes per wavenumber band and would again appear to satisfy the hypotheses necessary for Kolmogorov's theory to apply. Though our limiting model is undoubtedly intermittent, the energy spectrum may remain in accord with Kolmogorov's predictions.

Turbulence of course is only obtained from the Navier–Stokes equations in the presence of forcing and a small, but finite, dissipation. Nevertheless, the equilibrium statistical mechanics of a simple fluid, represented, for example, by all Fourier modes falling within a circle or a sphere, has provided useful insights into turbulent cascades in two and three dimensions.[11] In three dimensions the only relevant conserved quantity is the energy, which is quadratic in the velocity. The corresponding probability distribution assigns equal energy to each mode.[12] Our model system, in the absence of forcing and dissipation, has in addition to the energy a second isolating, extensive, conserved quantity that is cubic in the velocity and whose velocity derivatives generate the equations of motion. The equilibrium distribution established by our model is quite complicated and no longer yields equipartition of energy in all cases.

2. DEFINITION OF MODEL

Burgers' equation for a velocity $u(x, t)$ at position x and time t is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

where ν is the viscosity. The nonlinear term in (1) models the convective term in the Navier–Stokes equations. It conserves all integrals of the form

$$I_l = \int_0^{2\pi} u^l\,dx$$

provided u is periodic on the interval $(0, 2\pi)$. The energy E is just $\frac{1}{2}I_2$. The third invariant I_3 will henceforth be denoted by H because it generates the equations of motion for our model.

Equations of the same general form as (1) on the interval $(0, 2\pi)$ are conveniently written in Fourier space by specifying the interaction $T(l, m|-n) = T(m, l|-n)$ among a triad of wave numbers l, m, and n:

$$\partial u_n/\partial t = \sum_{l,m} T(l, m|-n)u_l u_m - \nu n^2 u_n \tag{2}$$

We will require that $T(l, m|n)$ be proportional to n. By translational invariance T must contain a delta-function factor, so we may define

$$T(l, m|n) = i(n/2)\,\delta(l + m + n)\,\Delta(l, m, n)$$

For Burgers' equation, $\Delta = 1$.

The nonlinear term in (2) conserves the energy $\frac{1}{2}\sum_n u_n u_{-n}$, provided

$$T(l, m|n) + T(l, n|m) + T(m, n|l) = 0$$

so we impose in addition that Δ is a symmetric function of its three arguments. The quantity

$$H = \sum_{n,m} u_n u_m u_{-n-m}\,\Delta(n, m, -n-m) \tag{3}$$

is conserved by (2) when $\nu = 0$ and Δ is symmetric. This is easily seen by using u_n and u_{-n} as conjugate variables in order to rewrite (2) as

$$\frac{\partial u_n}{\partial t} = -i\frac{n}{6}\frac{\partial H}{\delta u_{-n}} \tag{4}$$

Equations of the form (2) are conventionally cut off in Fourier space by requiring that $\Delta(l, m, n) = 1$ for

$$b^{-1} \leqslant |l/m| \leqslant b, \qquad b^{-1} \leqslant |n/m| \leqslant b, \qquad b^{-1} \leqslant |l/n| \leqslant b \tag{5}$$

and is zero otherwise.[13] The energy and H are of course still conserved because (5) preserves the symmetry of Δ in its three arguments.

If b approaches 2 in (5), the allowed interactions tend to a particularly simple form. The two modes whose wave vectors have magnitude n interact only with modes whose wave vectors have magnitude $\frac{1}{2}n$ or $2n$. Fourier space splits into decoupled chains of modes each labeled by an integer n_0 not divisible by 2. If we impose some cutoff Λ in Fourier space, only modes $u_{\pm n}$ with $n = n_0 2^l$ and $1 \leqslant l \leqslant L$ will interact. Using l to order these modes and rescaling, our model equations become, with $u_l = u^*_{-l}$,

$$\begin{aligned} du_1/dt &= 2iu_2u_1^* \\ du_l/dt &= 2^{l-1}i(u_{l-1}^2 + 2u_{l+1}u_l^*), \qquad 2 \leqslant l < L \\ du_L/dt &= 2^{L-1}iu_{L-1}^2 \end{aligned} \tag{6}$$

This system of equations has a considerable resemblance to other cascade models. The energy and H are still conserved.

It will prove convenient in the following sections to introduce three new sets of variables:

$$\begin{aligned} E_l &= \tfrac{1}{2}u_lu_{-l}, \qquad & 1 \leqslant l \leqslant L \\ H_l &= \mathrm{Re}(u_l^*u_{l-1}^2), \qquad & 2 \leqslant l \leqslant L \\ A_l &= \mathrm{Im}(u_l^*u_{l-1}^2), \qquad & 2 \leqslant l \leqslant L \end{aligned} \tag{7}$$

The first set is of course just the shell energies and the second set the shell "Hamiltonians," since (6) becomes with $H = \sum_{l=2}^{L} H_l$ just

$$du_l/dt = i2^l\, \delta H/\delta u_{-l} \tag{8}$$

The third set of variables A_l are related to the energy transfer between shells $\epsilon_l = -2^{l-1}A_l$, since the time rate of change of the energy may be written

$$dE_l/dt = \epsilon_l - \epsilon_{l+1}, \qquad 1 \leqslant l \leqslant L \tag{9}$$

with $\epsilon_1 = \epsilon_{L+1} = 0$. The quantity ϵ_l is just the energy current is wavenumber space and (9) is a discrete form of the one-dimensional continuity equation.

3. EQUILIBRIUM SOLUTIONS

For $L = 2$, (6) may be reduced to

$$-\frac{d}{dt}E_2 = \frac{d}{dt}E_1 = 2A_2, \qquad \frac{d}{dt}A_2 = 4E_1(-E_1 + 2E_2) \tag{10}$$

and a number of special solutions are easily obtained. For

$$H = \pm 2^{5/2}E^{3/2}/3^{3/2}$$

which are the upper and lower bounds on H, one finds

$$A_2 = 0, \qquad E_1 = 2E_2$$

$$u_1 = \pm(2\sqrt{E}/3)e^{\pm i\omega t}, \qquad u_2 = \pm(\sqrt{2}E/3)e^{\pm 2i\omega t}, \qquad \omega = 2\sqrt{2}E/3 \tag{11}$$

When $H = 0$, $iA_2 = u_2^*u_1^2$ and thus $|A_2|^2 = 8E_2E_1^2$. This allows a stationary

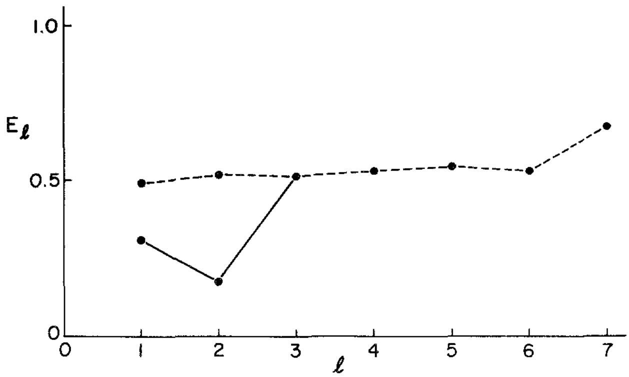

Fig. 1. Time-averaged energy spectra for H near zero. The solid line is for $L = 3$, $E = 1.0$, and $H = 4.5 \times 10^{-4}$. The dashed line is for $L = 7$, $E = 3.8$, and $H = 4.5 \times 10^{-2}$.

globally stable solution with $E_1 = 0$ and $E_2 = E$, as may be seen from (10). The system asymptotically approaches this state for any initial condition with $H = 0$. For solutions with H between these special values, E_1, E_2, and A_2 are periodic in time. The velocities themselves need not be periodic, because their absolute phases are not determined for given E_1, E_2, A_2, and H. Note the constraint $A^2 + H^2 = 8E_2E_1{}^2$.

Plasma physicists have studied a model, known as the three mode coupling problem, which reduces to (10) in a certain limit.[14] Their model also has a cubic invariant that generates the equations of motion.

Equations (6) were integrated forward from random initial conditions for a number of values of $L > 2$. The shell energies fluctuated in time but appeared to be statistically independent of the initial conditions for fixed E and H. Some time-averaged spectra for H near zero are plotted in Fig. 1.

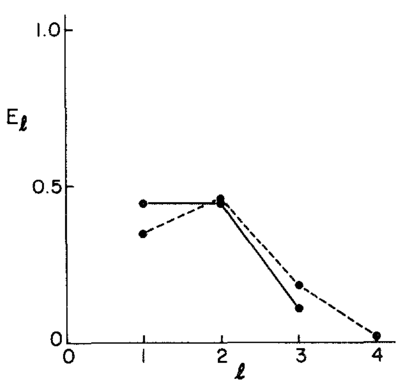

Fig. 2. Energy spectra for $H = H_{\max}$. The solid line is for $L = 3$, $E = 1.0$, and $H = H_{\max} = 1.26$. The dashed line is for $L = 4$, $E = 1.0$, and $H = H_{\max} = 1.28$.

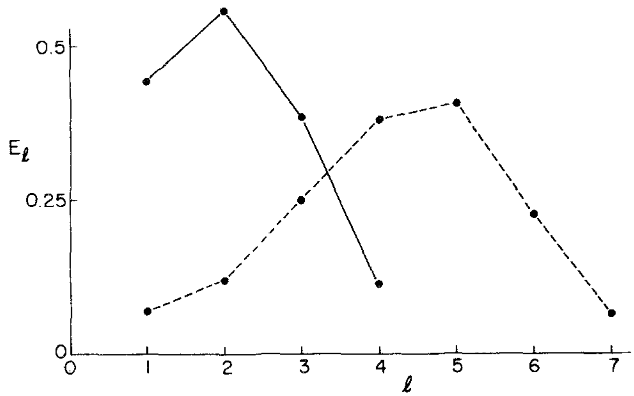

Fig. 3. Time-averaged spectra for H near H_{max}. Both curves are for $E = 1.5$ and $H = 2.0$. The solid line is for $L = 4$ and the dashed line is for $L = 7$.

This figure suggests that for H small there is a tendency toward equipartition as L grows; but even for $L = 7$, there is a concentration of energy in the large-wavenumber bands. Recall that for $L = 2$ and $H = 0$, all the energy is in the highest, $l = 2$, wavenumber band.

For fixed L and E, H is bounded above and below by $\pm H_{max}$. When H assumes its upper or lower bound, $u_l{}^* u_{l-1}^2$ is real, A_l is zero, and by (9), E_l is stationary. The maximum of H can be found from the extremum over all shell energies of

$$\bar{H} = 2^{2/3}(\sqrt{E_L} E_{L-1} + \cdots + \sqrt{E_2} E_1) \tag{12}$$

with the constraint $E = \sum_{l=1}^{L} E_l$. The values of E_l for $H = H_{max}$ appear unique and have been found explicitly for $L \leqslant 4$ (Fig. 2). For larger L we expect that the energy spectrum will peak at intermediate l and have tails toward the higher and lower wavenumber bands. The quantity $H_{max}/E^{3/2}$ is a bounded, monotone increasing function of L and must tend to a finite limit. The sequence appears to converge quite rapidly because from $L = 3$ to $L = 4$, $H_{max}/E^{3/2}$ increases from 1.26 to 1.28. For H near its maximum, the energy spectra obtained by time-averaging the solutions of (6) should approach the form expected for $H = H_{max}$. Figure 3 illustrates this behavior.

4. FORCED EQUATIONS WITH DISSIPATION

An inertial range, in which correlation functions scale with wavenumber k, is believed to occur in a turbulent fluid for $\lambda \ll k \ll \Lambda$, where λ and Λ are respectively the characteristic scales set by the forcing and the dissipation. The precise nature of the damping is believed to play no role in the inertial range provided it occurs only at high k. In order to obtain the maximum information from a limited range of scales a forcing term was added only to the first of Eqs. (6) and a phenomenological eddy damping was included in only the last equation. Thus

$$du_1/dt = \epsilon/u_1^* + 2iu_2u_1^* \tag{13a}$$

$$du_l/dt = 2^{l-1}i(u_{l-1}^2 + 2u_{l+1}u_l^*), \qquad 2 \leqslant l < L \tag{13b}$$

$$du_L/dt = 2^{L-1}i(u_{L-1}^2 + i2^{2/3}|u_L|u_L) \tag{13c}$$

The forcing term introduces energy at a constant rate ϵ. The eddy damping makes a negative-definite contribution to dE_L/dt. The factor $2^{2/3}$ in (13c) is partially conventional and its precise value has no appreciable effect on the energy cascade that develops when (13) is integrated forward in time. As written, (13) possesses a stationary "Kolmogorov" solution,

$$u_l = -i(2^{1/3}\epsilon)^{1/3}2^{-l/3} \tag{14}$$

so called because the energy transfer rate from band to band is a constant ϵ. Alternative methods for introducing both a driving force and dissipation into cascade models are discussed at length in Refs. 9 and 10.

Equations (13) were integrated numerically for $L = 8$ and 12. Correlation functions were computed as time averages and proved to be statistically stationary. A cascade developed in which energy was transferred to higher shells, the average energy transport between shells equalled ϵ, and the averaged shell energies had well-defined values. The "Kolmogorov" solution (14) is unstable and the energy transfer rate became temporally intermittent, that is, most of the transfer occurred during short periods of intense activity.

For L large, there is a hierarchy of equations in (13) that differ only in the factor 2^{l-1} that sets the frequency scale. Furthermore, because each mode is coupled only to its nearest neighbors and a statistically stationary though noisy state appears to exist, one might suspect that correlation functions for $1 \ll l \ll L$ would scale as some power of the frequency 2^l. Unfortunately, with only the order of ten levels, end effects are quite pronounced and exponents are difficult to infer. Of course the behavior of certain time averages, such as the average energy transfer $\langle\epsilon_l\rangle$, follows from only stationarity, that is, averaging (9),

$$\epsilon = \lim_{T\to\infty} \frac{1}{T}\int_0^T \epsilon_l(t)\, dt \tag{15}$$

No such argument exists for the average shell energies $\langle E_l\rangle$, though estimating the magnitude of the velocity by factoring (15) yields a "mean field" or "Kolmogorov" solution $\langle E_l\rangle \sim 2^{-2l/3}$.

In other cascade models temporal intermittency steepens the energy spectrum because bursts in u_l and u_{l-1} are correlated in time and hence (15) can be maintained with a smaller average energy.[9,10] Furthermore, the bursts become more intense and concentrated in time as l increases, thus steepening the spectrum. However, the numerical integrations of (13) did not show an energy spectrum any steeper than $2^{-2l/3}$. This probably reflects the presence of the second invariant H in (6). Defining $\gamma_l = -2^{l-1}\,\mathrm{Im}(u_l^* u_{l-1} u_{l-2}^2) = -2^{l-2}(H_l A_{l-1} + A_l H_{l-1})/E_{l-1}$, we find

$$dH_l/dt = \gamma_l - \gamma_{l+1}, \qquad 3 \leqslant l \leqslant L - 1 \tag{16}$$

Since $H_l(t)$ is statistically stationary, the average of the corresponding currents γ_l is independent of l. If $\langle \gamma_l \rangle \neq 0$, the energy exponent in the inertial range can again be estimated by factoring and averaging. One then finds $\langle E_l \rangle \sim 2^{-l/2}$.

The variance of the energy transfer $\langle \epsilon_l^2 \rangle$ is a useful quantitative measure of the level of intermittency. If we assume $\langle \epsilon_l^2 \rangle$ scales as $2^{\mu l}$, then our numerical results imply that μ is certainly positive and probably less than 1.0. End effects preclude any firmer conclusions. Note that for $L = 10$, (13) already spans a range of 10^3 in frequency; it is not a simple matter to double L.

5. DISCUSSION

A simple model of the turbulent energy cascade has been derived with a number of novel properties attributable to a cubic, extensive, and inviscid constant of the motion. One is reminded of two-dimensional turbulence, where there is also an extensive invariant in addition to the energy, the enstrophy. It is well known that because the ratio of the energy and the enstrophy is always the squared wave vector, a simultaneous, statistically steady cascade of both cannot exist.[15] One observes instead two separate cascades in opposite directions in Fourier space.[11] Our cubic invariant is a consequence of the cubic invariant in the one-dimensional Burgers' equation and is not expected to occur in truncations of the Navier–Stokes equations in higher dimensions.[9],[4]

The precise impact of the cubic invariant on the cascade solutions of our forced model, and by implication also on Burgers' equation, is not yet clear. In particular, $\langle H_l \rangle$ was not computed for the forced model. The contri-

[4] In Ref. 16, Hald has derived several low-order truncations to the two-dimensional Euler equation for which there are a number of invariants in addition to the energy and the enstrophy. Of course any system of n first-order equations can have up to n invariants. The invariants he has found are not obviously generalizable to larger systems and are presumably not extensive in the sense that their value in a system is the sum of their values for a partition of that system, neglecting end effects.[17] An extensive cubic invariant has been found in another one-dimensional model by Fyfe and Montgomery.[18]

bution of the forcing term to dH_2/dt is not of definite sign, yet it is not clear whether, on the average, it might still inject H "stuff" into the cascade. Even if $\langle \gamma_l \rangle = 0$, the implied constraint could still affect the cascade. It was inferred that the cubic invariant flattens the energy spectrum of our model system. By continuity, it should have the same effect on the cutoff Burgers' equations for $b \gtrsim 2$. Thus, in a simulation for $b \gtrsim 2$, a Kolmogorov-like energy spectrum might be found to within sampling and numerical errors, even though, as measured by the variance of the energy transfer, the system was intermittent. When b is increased and Burgers' equation is recovered, the cubic invariant, though formally still present, may have no effect on the cascade since H is no longer extensive. That is, because H can no longer be written as a sum of terms H_l each containing factors from only a few adjacent wavenumber bands, the constancy of the flux of H in Fourier space is less likely to constrain the cascade.

The equilibrium solutions of our model are also of some interest. For H near its upper and lower bounds, equipartition of energy is not expected to occur even for large L. However, if one imagines building up a large-L model by coupling together a sequence of smaller models, both H and E will scale as L. However, we showed that $H_{\max}/E^{3/2}$ tends to a finite limit and thus $H/H_{\max} \sim L^{-1/2}$. Quite possibly in this situation equipartition of energy is again obtained.

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